Introduction to implicit automata in typed λ-calculi and higher-order transducers

Lê Thành Dũng (Tito) Nguyễn (CNRS & Aix-Marseille Univ.)
based on joint work with several people, mainly **Cécilia Pradic** (*looking for a new job!*)

Higher-Order Computation in Implicit and Descriptive Complexity
26-27 May 2026, IRIF, Paris

Two convergent lines of work about *relating the expressive power of automata-based and λ-calculus-based formalisms*

- **Implicit automata:** motivated by questions internal to λ-calculus, and by Implicit Computational Complexity (ICC)

Template for ICC theorems

The languages/functions computed by programs of type T in the programming language $\mathcal{P}$ are exactly those in the complexity class $\mathcal{C} \in \{\mathsf{P}, \mathsf{NP}, \mathsf{PSPACE}, \dots\}$.

- Active research field since the 1990s, cf. Péchoux's HDR
- Historical example (Girard): $\mathcal{P}$ = Light Linear Logic, $\mathcal{C} = \mathsf{P}$ (polynomial time)
- Basic idea here: $\mathcal{C} \in$ automata theory

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- **Higher-order grammars and transducers:** applying λ-calculus to motivations from automata and formal languages
  - older tradition (Damm / Engelfriet & Vogler, 1980s)
  - 21st c. revival (Kanazawa '08; Gallot, Lemay & Salvati '20; ...): linear types

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What about the simply typed λ-calculus?

- Hillebrand, Kanellakis & Mairson 1993: ELEMENTARY
  - "matches" TOWER-completeness of normalization [Statman 1979; Schmitz 2016]
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- What about the standard *Church encodings?*

Simply typed λ-terms operating on Church encodings

Some results known, e.g. Leivant 1993, Zaionc 1994

Theorem (Schwichtenberg 1975)

The functions $\mathbb{N}^k \to \mathbb{N}$ *definable by simply-typed* λ*-terms* $t : \mathsf{Nat} \to \cdots \to \mathsf{Nat} \to \mathsf{Nat}$ *are the extended polynomials* (*generated by 0, 1,* $+$, $\times$, `id` *and* `ifzero`).

where Nat is the type of *Church numerals*: $\mathsf{Nat} = (o \to o) \to o \to o$

$$n \in \mathbb{N} \quad \rightsquigarrow \quad \overline{n} = \lambda f.\, \lambda x.\, f\,(\ldots\ (f\,x)\ldots) : \mathsf{Nat} \text{ with } n \text{ times } f$$

All inhabitants of Nat are equal to some $\overline{n}$ up to $=_{\beta\eta}$

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Let's add a bit of (meta-level) polymorphism: $\overline{n} : \mathsf{Nat}[A] = \mathsf{Nat}[A/o]$ for $n \in \mathbb{N}$

More difficult question (what is the right perspective on it?)

Choose some simple type A and some term $t : \mathsf{Nat}[A] \to \mathsf{Nat}$.
What functions $\mathbb{N} \to \mathbb{N}$ can be defined this way?

Defining languages in the simply typed λ-calculus

Church encodings of binary strings [Böhm & Berarducci 1985]

$\simeq$ `fold_right` on a list of characters (generalizable to any alphabet; $\mathsf{Nat} = \mathsf{Str}_{\{1\}}$):

$$\overline{\texttt{011}} = \lambda f_0.\ \lambda f_1.\ \lambda x.\ f_0\ (f_1\ (f_1\ x)) : \mathsf{Str}_{\{\texttt{0},\texttt{1}\}} = (o \to o) \to (o \to o) \to o \to o$$

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Example: $t = \lambda s.\ s\ \texttt{id not true} : \mathsf{Str}_{\{\texttt{0},\texttt{1}\}}[\mathsf{Bool}] \to \mathsf{Bool}$ (even number of `1`s)

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Theorem (Hillebrand & Kanellakis 1996)

All regular languages, *and only those, can be defined this way.*

Theorem (Hillebrand & Kanellakis, LICS'96)

For any type A and any simply typed λ-term $t : \mathsf{Str}_\Sigma[A] \to \mathsf{Bool}$, the language $\mathcal{L}(t) = \{w \in \Sigma^* \mid t\,\overline{w} \to_\beta^* \mathtt{true}\}$ is regular.

Part 1 of proof.

Fix type A. Any *denotational semantics* $[\![-]\!]$ quotients words:

$$w \in \Sigma^* \rightsquigarrow \overline{w} : \mathsf{Str}[A] \rightsquigarrow [\![\overline{w}]\!]_{\mathsf{Str}_\Sigma[A]} \in [\![\mathsf{Str}_\Sigma[A]]\!]$$

$[\![\overline{w}]\!]_{\mathsf{Str}_\Sigma[A]}$ determines behavior of w w.r.t. all $\mathsf{Str}_\Sigma[A] \to \mathsf{Bool}$ terms:

$$w \in \mathcal{L}(t) \iff t\,\overline{w} \to_\beta^* \mathtt{true} \underbrace{\iff}_{\text{assuming } [\![\mathtt{true}]\!] \neq [\![\mathtt{false}]\!]} [\![t\,\overline{w}]\!] = [\![t]\!]([\![\overline{w}]\!]) = [\![\mathtt{true}]\!]$$

Goal: to decide $\mathcal{L}(t)$, compute $w \mapsto [\![\overline{w}]\!]$ in some denotational model.

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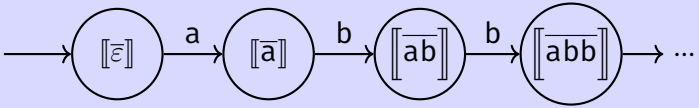
Part 2 of proof.

We use $[\![-]\!] : \mathrm{ST}\lambda\mathrm{C} \to \mathsf{FinSet}$ to build a DFA with states $Q = [\![\mathsf{Str}_\Sigma[A]]\!]$, acceptation as $[\![t]\!](-) = [\![\mathtt{true}]\!]$.



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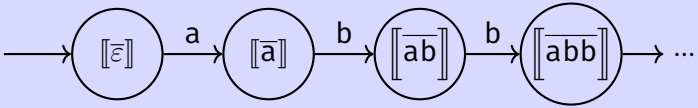

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Usual transduction classes (functions $\Gamma^* \to \Sigma^*$)

What is the right generalization of regular languages to *transductions*?

There are several canonical ones!

$$\underbrace{\text{sequential fonctions}}_{\substack{\textit{deterministic} \text{ finite transducers}}} \subsetneq \underbrace{\text{rational functions}}_{\textit{nondeterministic} \text{ transducers}} \subsetneq \text{regular functions}$$

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All **regularity reflecting:** L regular $\implies f^{-1}(L)$ regular

Corollary of the Hillebrand–Kanellakis theorem

Functions defined by $\mathsf{Str}_\Gamma[A] \to \mathsf{Str}_\Sigma$ in the simply typed λ-calculus are also regularity reflecing.

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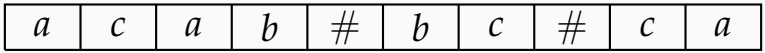
We shall be interested in *regular functions*; many definitions,

such as *streaming string transducers*

Streaming string transducers [Alur & Černý 2010] a.k.a. register transducers

Deterministic finite state automaton + string-valued *registers*. Example:

$$\begin{array}{lrcl} \texttt{mapReverse}: & \{a,b,c,\#\}^* & \to & \{a,b,c,\#\}^* \\ & w_1\#\ldots\#w_n & \mapsto & \texttt{reverse}(w_1)\#\ldots\#\texttt{reverse}(w_n) \end{array}$$



$$X = \varepsilon \qquad Y = \varepsilon$$

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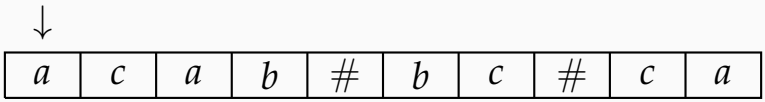

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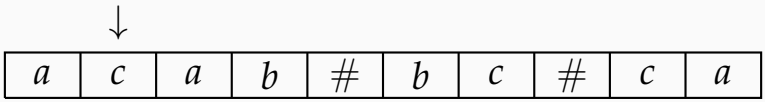

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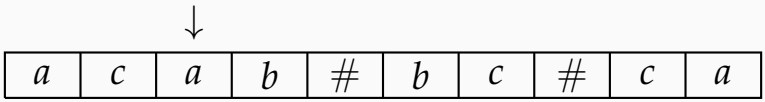

$$X = aca \qquad Y = \varepsilon$$

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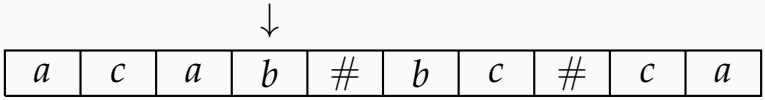

$$X = baca \qquad Y = \varepsilon$$

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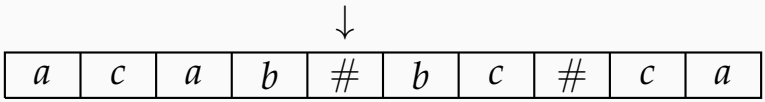

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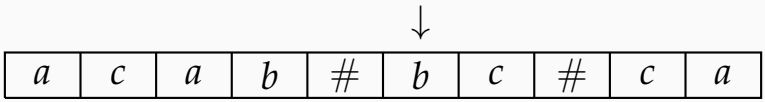

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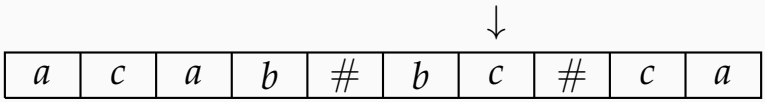

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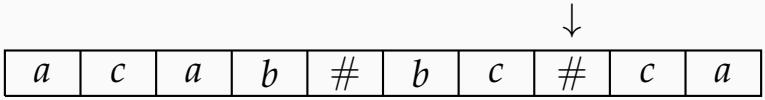

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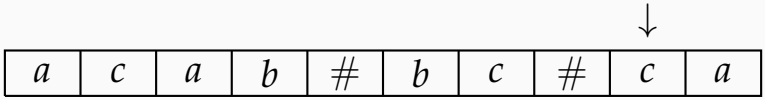

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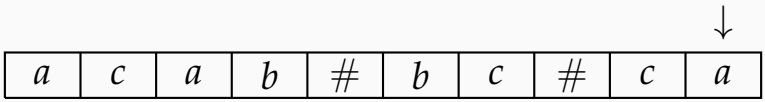

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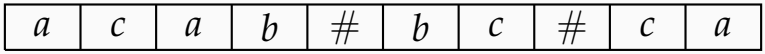

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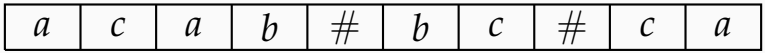

$X = ac \qquad Y = baca\#cb\# \qquad \texttt{mapReverse}(\ldots) = YX = baca\#cb\#ac$

Regular functions = computed by copyless SSTs

$$a \mapsto \begin{cases} X := aX \\ Y := Y \end{cases} \qquad \# \mapsto \begin{cases} X := \varepsilon \\ Y := YX\# \end{cases}$$

each register appears at most once on the right of a $:=$ in a transition

Linearity

Regular functions = computed by *copyless* streaming string transducers
Restrictions on "copying power": old theme in automaton theory

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λ-calculus counterpart: linear types (Girard 1987)

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$$A, B ::= o \mid \overbrace{\underbrace{A \multimap B}_{\text{linear functions}} \mid A \otimes B}^{\text{multiplicatives}} \mid \overbrace{A \,\&\, B \mid A \oplus B}^{\text{additives}} \mid \underbrace{A \to B}_{\text{non-linear functions}}$$

First steps in implicit transducers

Linear Church encodings

$\mathsf{Str}_{\{a,b\}} = (o \multimap o) \to (o \multimap o) \to (o \multimap o)$, *mutatis mutandis* for Str_Σ

Definition: a type of the $\lambda\ell^{\oplus\&}$-calculus is *purely linear* if it contains no "$\to$"

Theorem (N. & Pradic; proof technique on next slides)

$f : \Gamma^* \to \Sigma^*$ *is* regular $\iff$ $\exists\, a$ purely linear *type* A *and* $t : \mathsf{Str}_\Gamma[A] \multimap \mathsf{Str}_\Sigma$ *in the* $\lambda\ell^{\oplus\&}$*-calculus such that* $\forall w \in \Gamma^*,\ \overline{f(w)} =_\beta t\ \overline{w}$

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Works also for regular *tree-to-tree* functions: $\mathsf{Tree} = (o \multimap o \multimap o) \to \cdots \to o$

- Additives $\&/\oplus$ required to get enough expressive power in tree-to-tree case
- String-to-string without $\&/\oplus$: use *affine* types + Krohn-Rhodes decomposition

Proof of $\lambda\ell^{\oplus\&}$-definable $\implies$ regular function

1. By syntactic analysis,[1] turn $t : \mathsf{Str}_\Gamma[A] \multimap \mathsf{Str}_\Sigma$ into a "$\lambda\ell^{\oplus\&}$-transducer"
   - add constant symbols $\varepsilon : o$ and $a : o \multimap o$ for $a \in \Sigma$
   - λ-terms defining a one-way computation:

$$u : A \qquad (t_a : A \multimap A)_{a\in\Gamma} \qquad v : A \multimap o$$

[1] Easy with just $\to$ / $\multimap$ /&, uses focusing with $\otimes/\oplus$.

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$\lambda\ell^{\oplus\&}$-transducers $=$ variant of *higher-order transducers*

2. Compile $\lambda\ell^{\oplus\&}$-transducer to streaming string trans. by semantic evaluation
   - Monoidal closed category tailored for this purpose:
     Dialectica-like bicompletion + functorial automata + string combinatorics

[1] Easy with just $\to$ / $\multimap$ /&, uses focusing with $\otimes/\oplus$.

Alternative proof of affine λ-definable $\Longrightarrow$ regular function

This time **we do not have the additives** $\oplus/\&$

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   - categorical GoI [Pradic & Price '24], following earlier $\mathsf{Int}(\mathcal{C})$ vs automata works [Hines '03, Katsumata '08]

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Over trees: affine λ-trans. $\rightsquigarrow$ tree-walking transducers

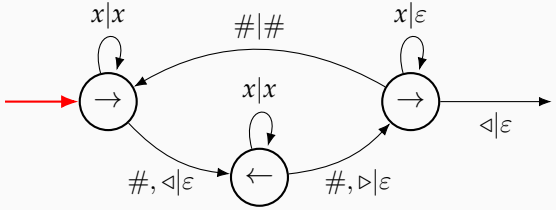
Theorem (Bojańczyk & Colcombet 2005)

Tree-walking automata $\subsetneq$ regular tree languages

Corollary: affine λ-transducers $\subsetneq$ regular tree functions

Two-way transducers characterize regular functions

Example: $w_1\# \ldots \#w_n \longmapsto w_1 \cdot \texttt{reverse}(w_1)\# \ldots \#w_n \cdot \texttt{reverse}(w_n)$



$(x \in \{a, b, c\})$

▷	a	b	c	#	b	a	c	#	c	b	◁

Output:

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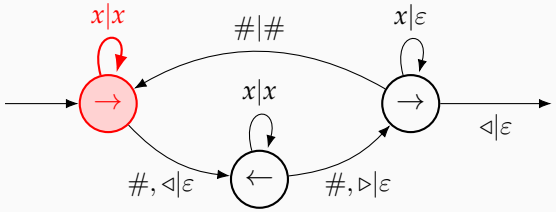

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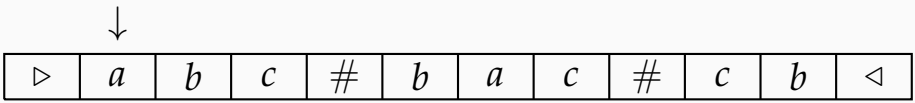

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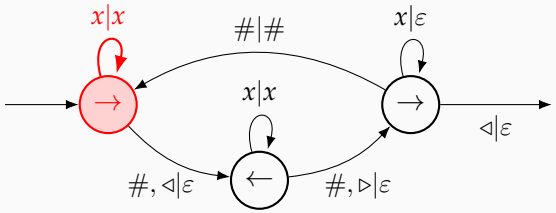

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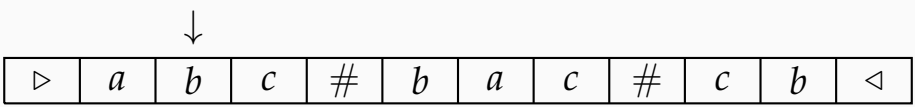

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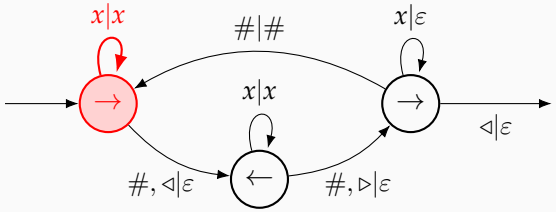

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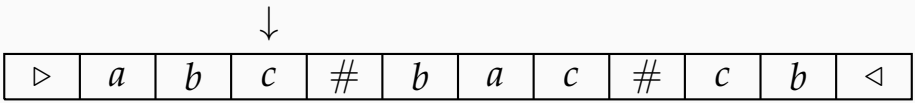

Output: ab

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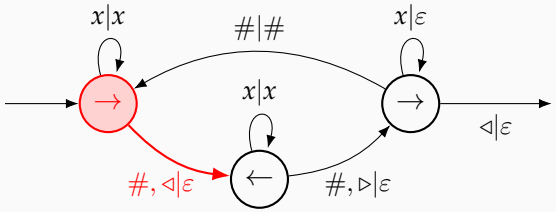

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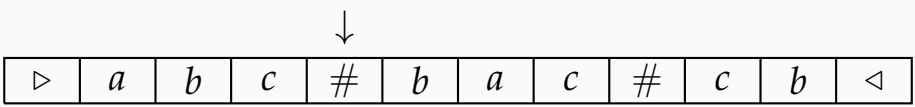

Output: abc

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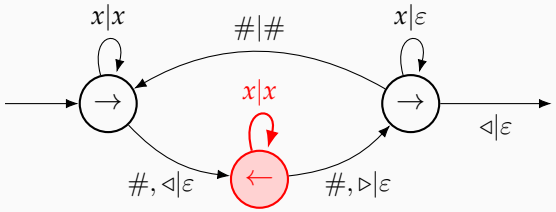

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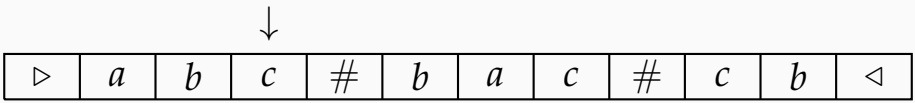

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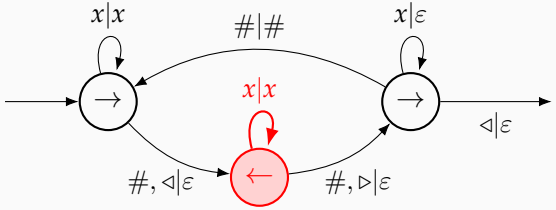

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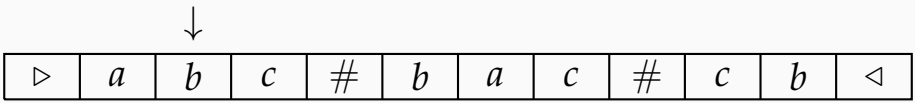

Output: *abcc*

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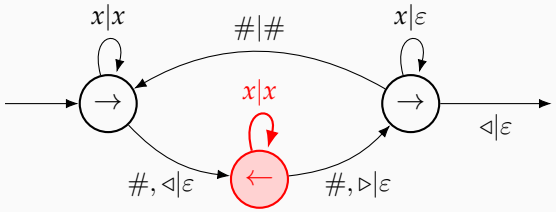

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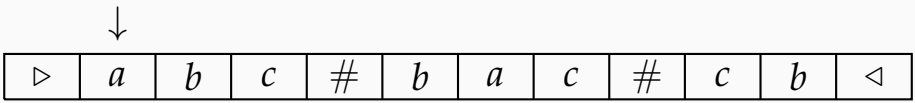

Output: *abccb*

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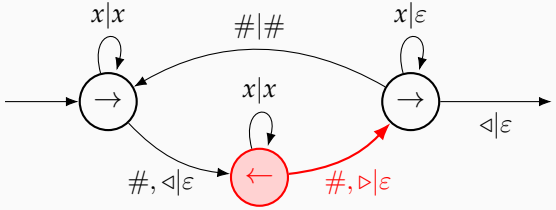

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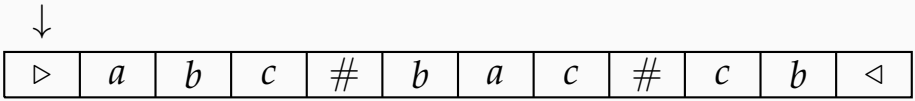

Output: *abccba*

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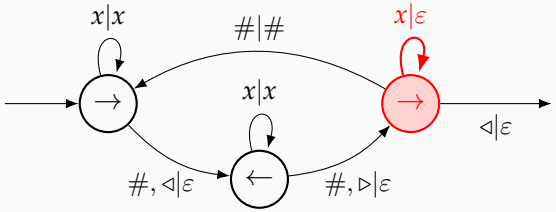

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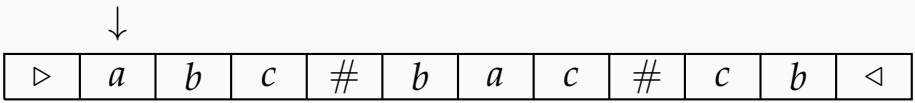

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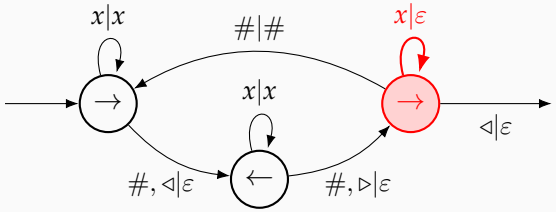

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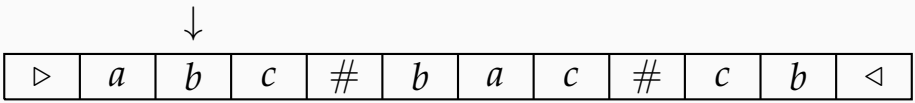

Output: *abccba*

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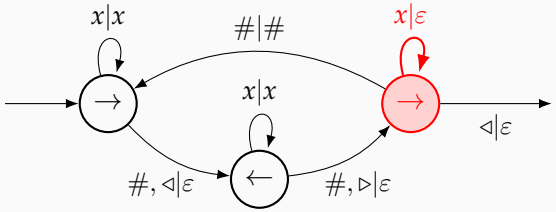

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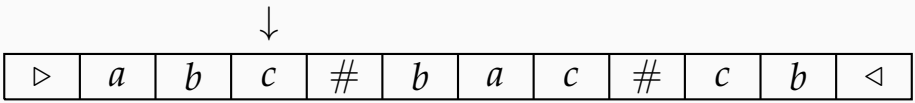

Output: *abccbaa*

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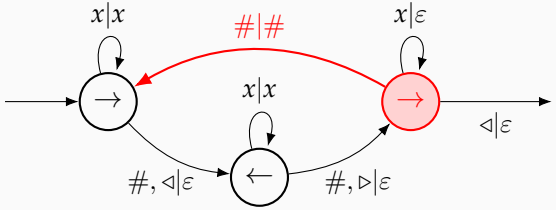

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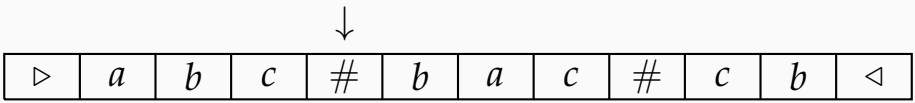

Output: *abccba*

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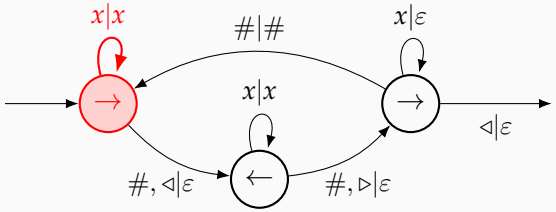

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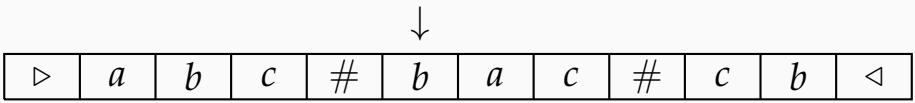

Output: $abccba\#$

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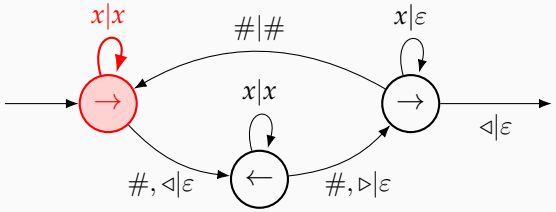

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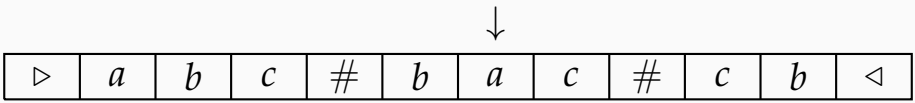

Output: $abccba\#b$

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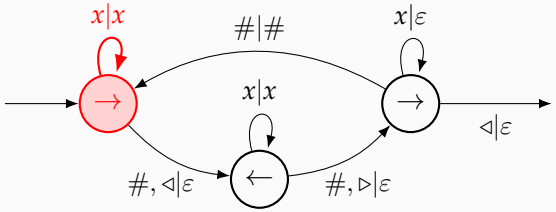

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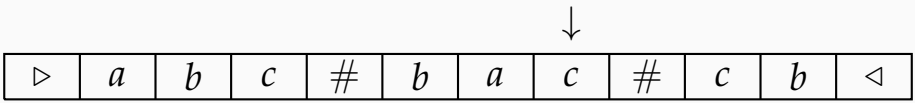

Output: $abccba\#ba$

Two-way transducers characterize regular functions

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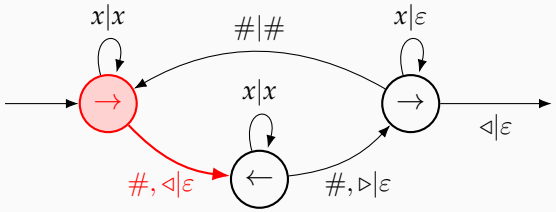

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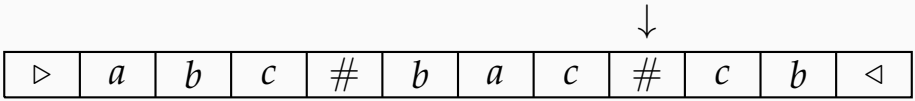

Output: $abccba\#bac$

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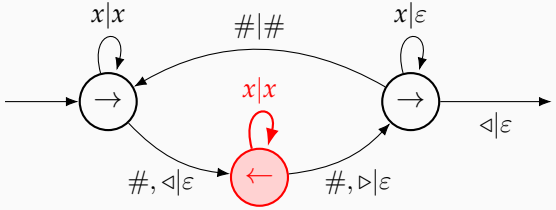

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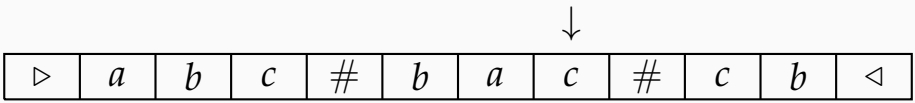
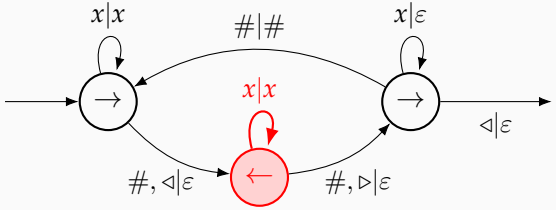


Output: $abccba\#bac$

Two-way transducers characterize regular functions

Example: $w_1\# \dots \# w_n \longmapsto w_1 \cdot \texttt{reverse}(w_1)\# \dots \# w_n \cdot \texttt{reverse}(w_n)$



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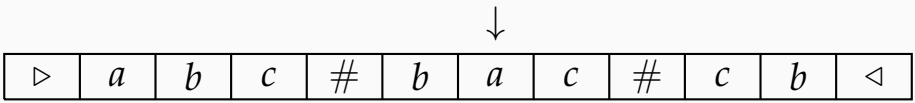

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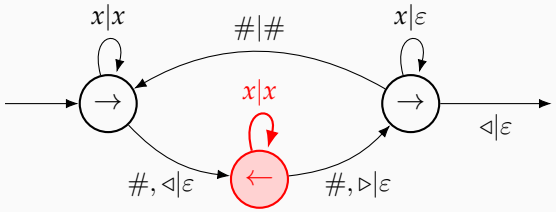

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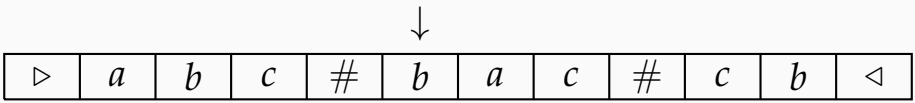

Output: *abccba#bacca*

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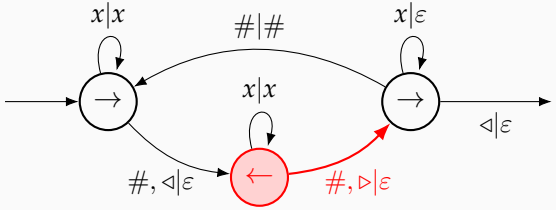

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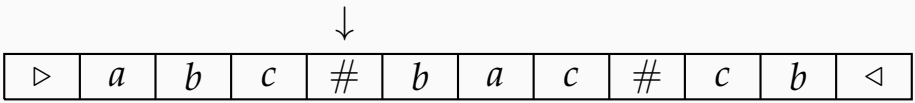

Output: *abccba#baccab*

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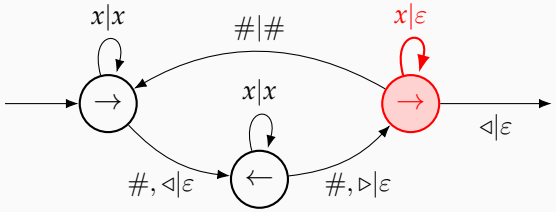

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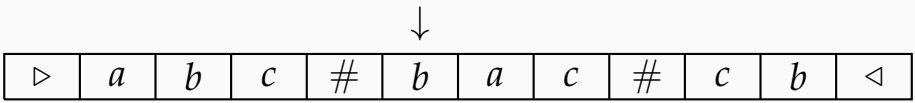

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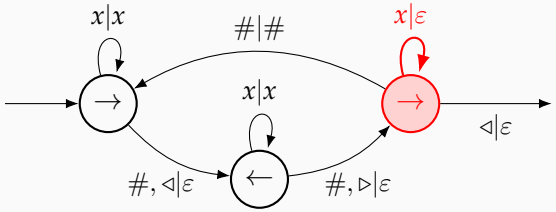

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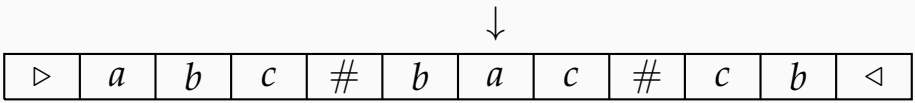

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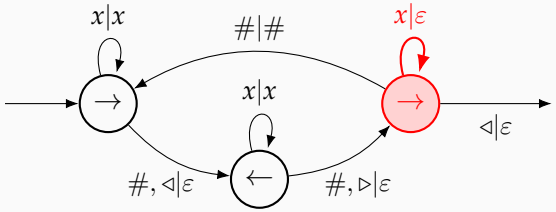

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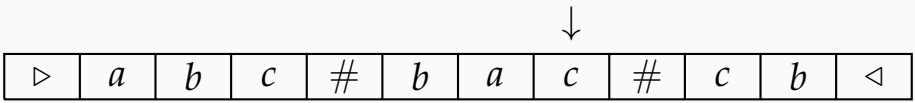

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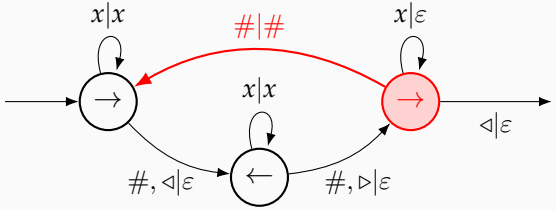

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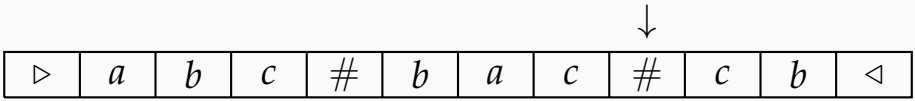

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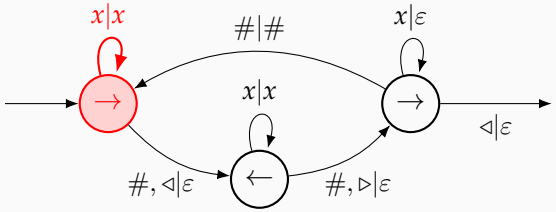

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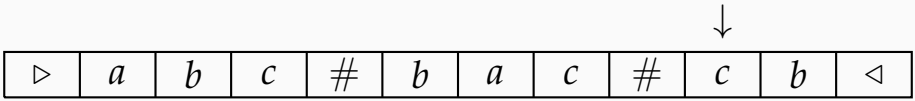

Output: $abccba\#baccab\#$

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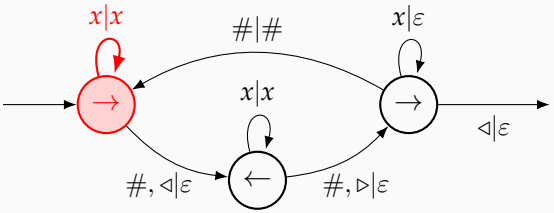

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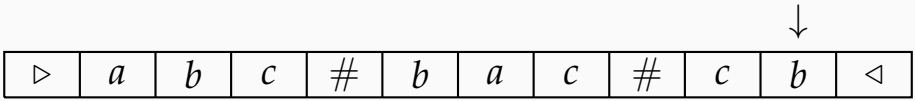

Output: $abccba\#baccab\#c$

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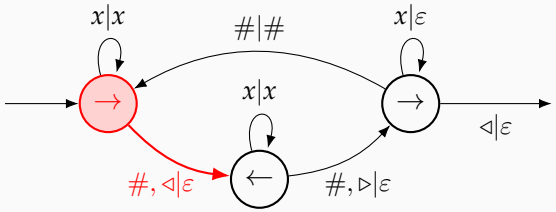

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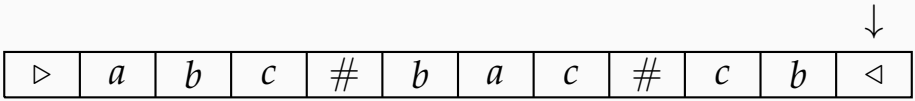

Output: *abccba#baccab#cb*

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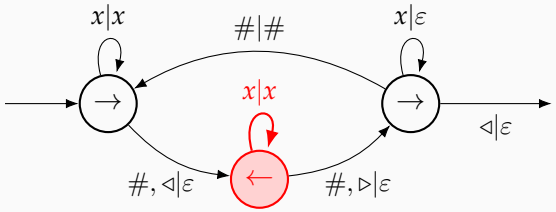

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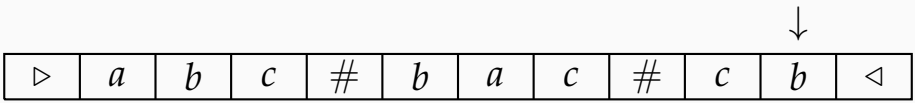

Output: $abccba\#baccab\#cb$

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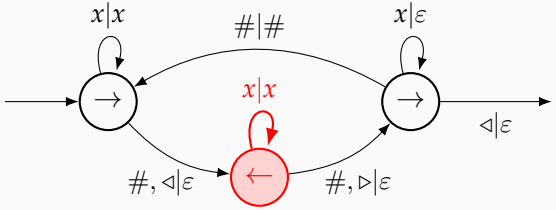

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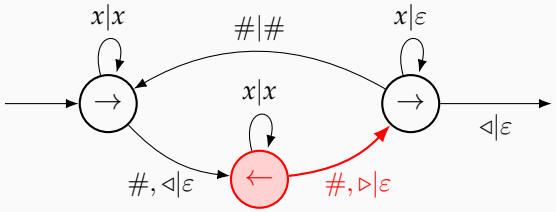

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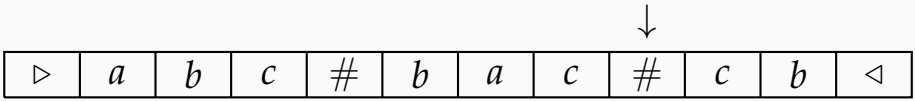

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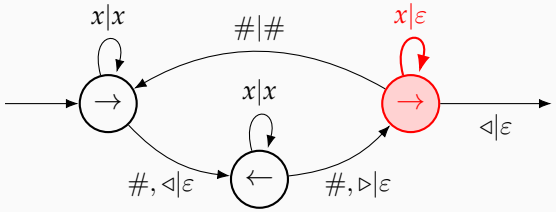

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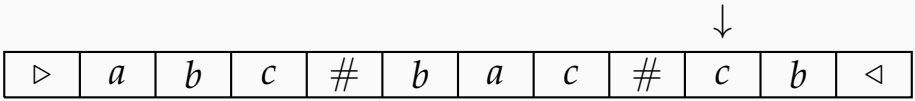

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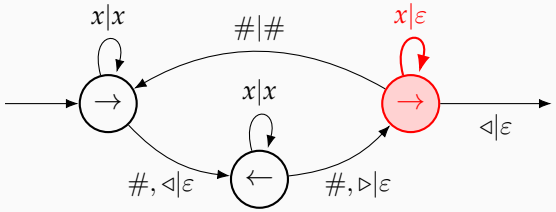

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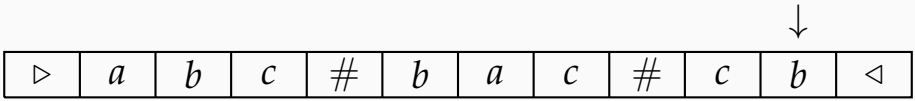

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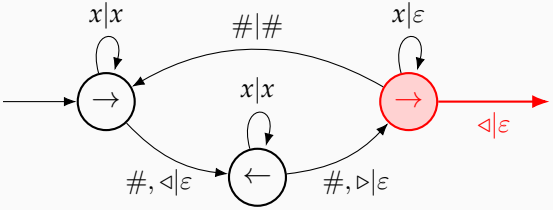

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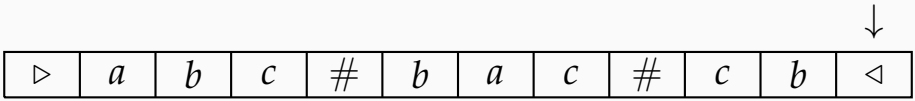

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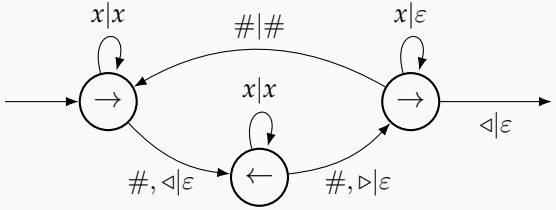

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▷	a	b	c	#	b	a	c	#	c	b	◁

Output: $abccba\#baccab\#cbbc$

λ-transducers with regular lookaround

Corollary of Bojańczyk & Colcombet 2005

Affine tree-to-tree λ-transducers $\subsetneq$ regular tree functions.

- Previously: to get $=$, use additives $\oplus/\&$

λ-transducers with regular lookaround

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- Alternatively: query regular properties of input during computation

Theorem (Gallot, Lemay & Salvati 2021)

Affine tree-to-tree λ-trans. with regular lookaround $=$ *regular tree functions.*

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The *exponential modality* '!' of linear logic makes a type duplicable

Slightly non-linear λ-transducers with regular lookaround [N. & Vanoni 2025]

Relating λ-transducers of low !-depth to automata models,
e.g. "invisible pebble tree transducers"

Proofs: via Interaction Abstract Machine

Higher-order transducers vs implicit automata

Pros and cons of λ-transducers:

- Easy to add automata-theoretic features, e.g. regular lookaround
- Only traverse their input once

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Theorem (N. & Pradic — note the $\to$ instead of $\multimap$)

$$f : \Gamma^* \to \Sigma^* \textit{ is } \text{comparison-free polyregular}$$
$$\iff$$
$$\exists\, a \text{ purely linear } \textit{type } A \textit{ and } t : \mathsf{Str}_\Gamma[A] \to \mathsf{Str}_\Sigma \textit{ in the } \lambda\ell^{\oplus\&}\textit{-calculus}$$
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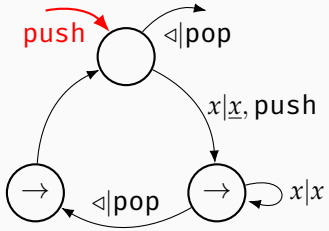
$\rightsquigarrow$ *Discovery* of the "comparison-free" subclass of Bojańczyk's *polyregular functions*, and of its good properties (e.g. closure under composition)

Comparison-free polyregular functions, a.k.a. polyblind functions

Example: "comparison-free squaring"

$\texttt{cfsquaring}(abb) = \underline{a}abb\underline{b}abb\underline{b}abb$



▷	a	b	c	◁

output =

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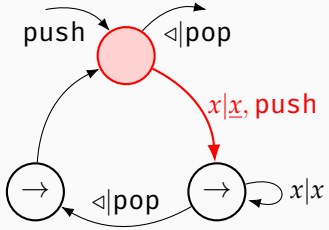

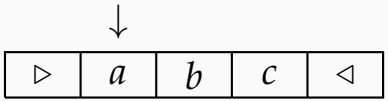

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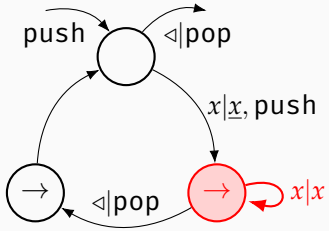

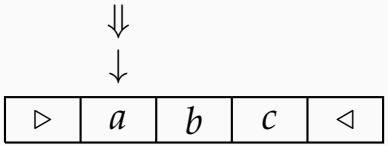

output $= \underline{a}$

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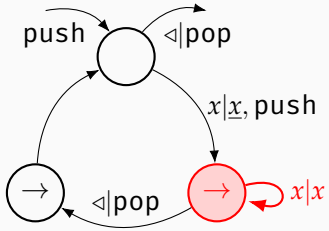

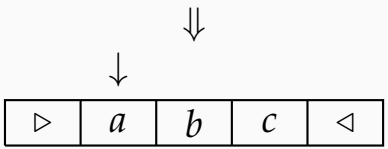

output $= \underline{a}a$

Comparison-free polyregular functions, a.k.a. polyblind functions

Example: "comparison-free squaring"

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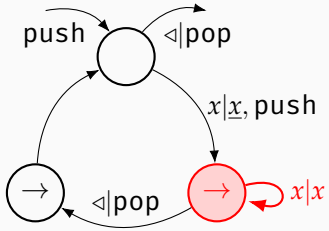

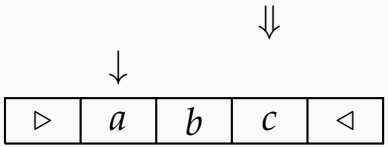

output $= \underline{a}ab$

Comparison-free polyregular functions, a.k.a. polyblind functions

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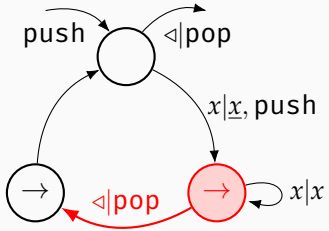

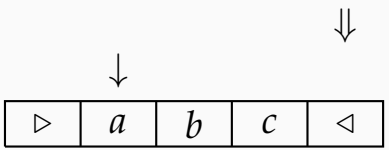

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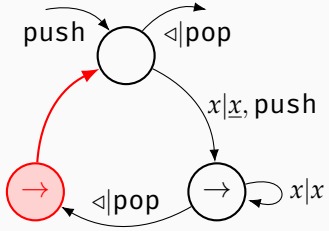

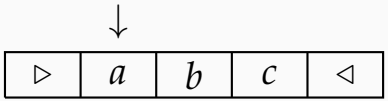

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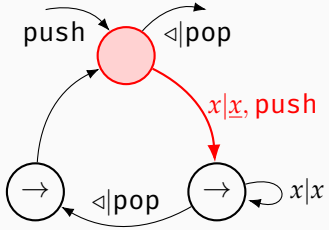

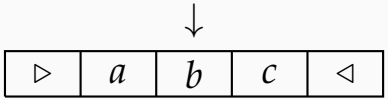

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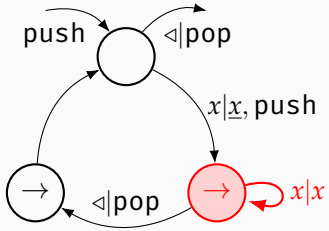

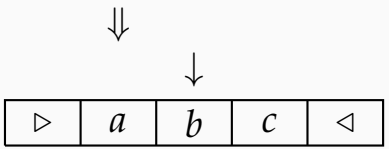

output $= \underline{a}abc\underline{b}$

Comparison-free polyregular functions, a.k.a. polyblind functions

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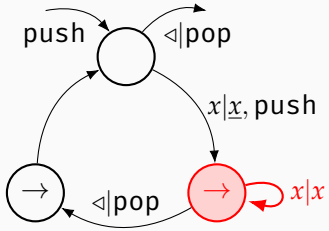

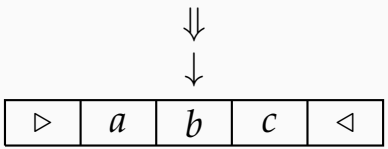

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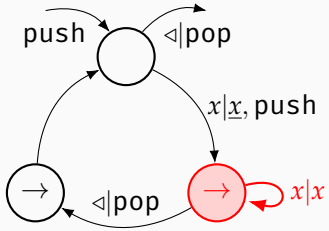

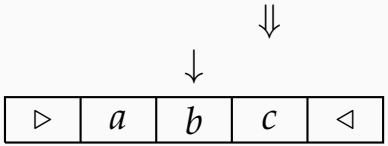

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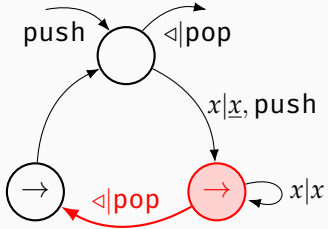

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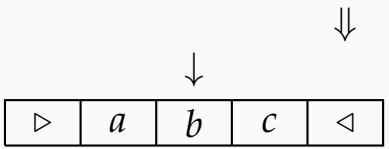

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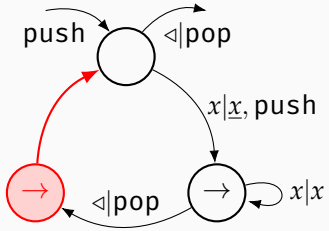

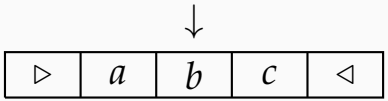

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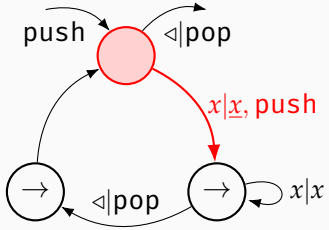

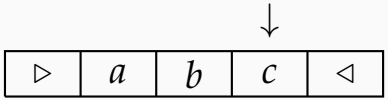

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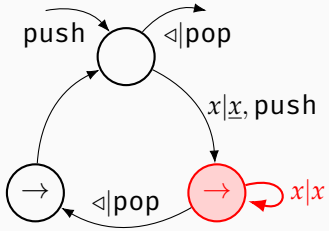

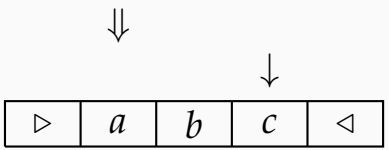

output $= \underline{a}abc\underline{b}abc\underline{c}$

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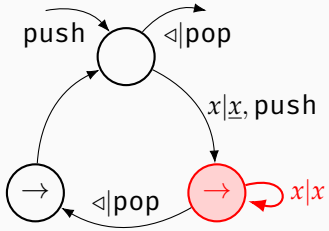

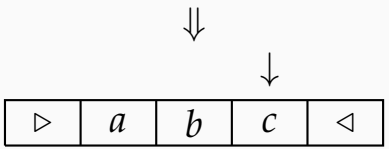

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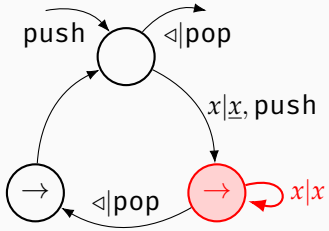

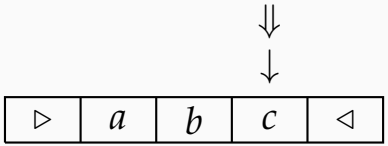

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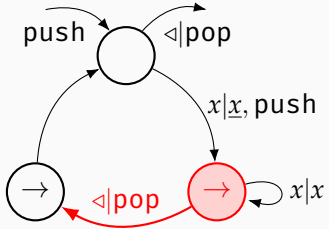

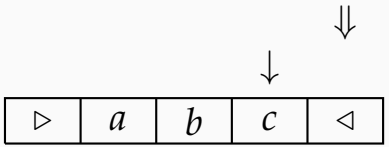

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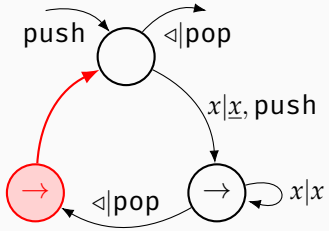

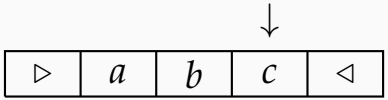

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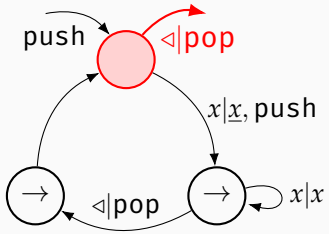

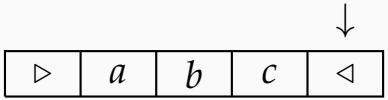

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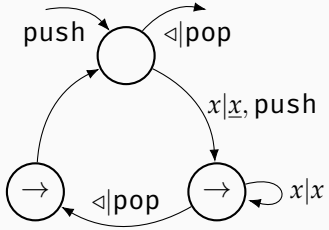

▷	a	b	c	◁

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Non-misery of implicit automata??

Good news: we've worked with "natural" typed λ-calculi for now

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Next: ideas from λ-calculus help solve an open problem in automata

The power of additive branching

Hennie machines = *bounded-visit* two-way transducers that can write on input cells

Theorem (Dartois, N., Peyrat '26)

The following define the same functions:

- $t : \mathsf{Tree}^{\otimes}[A] \multimap \mathsf{Tree}^{\&}$, A *$\to$-free*
- *tree-to-tree* Hennie machines.

$$\mathsf{Tree}^{\otimes} = (o \multimap o \multimap o) \to \cdots \to o$$

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Theorem (Clairambault, Murawski '19)

The following define the same infinite trees:

- *affine* higher-order recursion schemes *with additive branching;*
- *tree stack automata.*

Morally same result, via close analogies

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Proofs: continuation-passing style encodings / memoryful GoI with additives

one is a straightfoward transposition of the other

Consequences for automata theory

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- Hard application:

Theorem (Colcombet, Lhote & Ohlmann '26 — first conjectured by Bárány '07)

Every automatic ω-word has a decidable MSO theory.

More results

- *non-commutative* affine λ-calculus $\rightsquigarrow$ star-free languages [N. & Pradic '20]
- simply typed (light|parsimonious) affine λ-calculus $\rightsquigarrow$ polyregular functions [Wojciech Paupa's master thesis, Univ. Warsaw, to appear]

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Simply typed λ-calculus / unsafe higher-order transducers: open questions

- Reflection of regular cost functions?
- Deciding asymptotic growth? Polynomial growth implies polyregular?
- Study of order-k output languages (motivation: [Kiefer, N., Pradic '23])
  $\rightsquigarrow$ use a $\{0, 1, \infty\}$-graded Taylor expansion with remainder of λ-terms?

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safe order-k transducers $=$ k-iterated pushdown transducers

[Engelfriet & Vogler 1988] $=$ (order-1 a.k.a. macro tree transducers)^k

Simply typed λ-calculus / unsafe higher-order transducers: open questions

- Reflection of regular cost functions?
- Deciding asymptotic growth? Polynomial growth implies polyregular?
- Study of order-k output languages (motivation: [Kiefer, N., Pradic '23])
  $\rightsquigarrow$ use a $\{0, 1, \infty\}$-graded Taylor expansion with remainder of λ-terms?